Memorandum

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Von: Andreas Suter
Telefon: +41 (0)56 310 4238
Raum: WLGA / 119
e-mail: andreas.suter@psi.ch,

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Four-Point Resistivity Correction Factors for Thin Films: Addendum and Corrections

This memo is an extension and correction of the memo form B.M. Wojek entitled "*Four-point resistivity correction factors for thin films*" from January 5, 2009. It tries to give a more general view and corrects for errors found in the mentioned memo. The motivation was to get a proper understanding which in turned was used to write a little ROOT/C++ class which can be used to calculate the correction factors for any arbitrary four-point geometry on a square platelet. As described below.

The starting point of the discussion is the following formula (for references see B.M. Wojek's memo):

$$\Delta V = V_2 - V_3 = \left(\frac{\rho I}{2\pi}\right) \int_0^\infty \{J_0(kr_{21}) - J_0(kr_{24}) - J_0(kr_{31}) + J_0(kr_{34})\} \cdot \frac{\cosh(kt)}{\sinh(kt)}\, \mathrm{d}k, \qquad (1)$$

where the geometry is shown in Fig.1. $J_0(x)$ are 0^{th} order Bessel functions[1], and $r_{ij} = |\vec{r}_i - \vec{r}_j|$

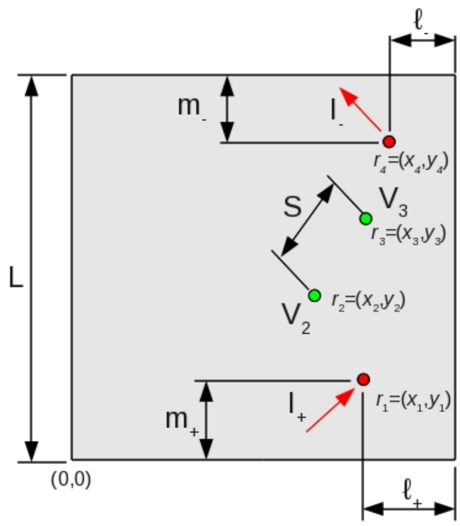

Figure 1: four-point geometry

In the following everything will be written in normalized variables. The normalization length will be the distance between V_2 and V_3 and be called

$$s = |\vec{r}_3 - \vec{r}_2|$$

[1] *not* spherical Bessel functions!

using this the following new variables will be introduced:

$$\begin{aligned} R_{ij} &= r_{ij}/s \\ \tau &= t/s \\ \kappa &= k \cdot s \implies \mathrm{d}k = \frac{1}{s}\,\mathrm{d}\kappa \end{aligned}$$

$$\Delta V = V_2 - V_3 = \left(\frac{\rho I}{2\pi s}\right) f(R_{ij}, \tau), \tag{2}$$

where

$$f(R_{ij}, \tau) = \int_0^\infty \{J_0(\kappa R_{21}) - J_0(\kappa R_{24}) - J_0(\kappa R_{31}) + J_0(\kappa R_{34})\} \cdot \frac{\cosh(\kappa\tau)}{\sinh(\kappa\tau)}\,\mathrm{d}\kappa = f_1(R_{ij}, \tau) + f_2(R_{ij}, \tau) \tag{3}$$

where the additional functions $f_k(R_{ij}, \tau)$ are defined as

$$\begin{aligned} f_1(R_{ij}, \tau) &= \int_0^\infty \{J_0(\kappa R_{21}) - J_0(\kappa R_{24}) - J_0(\kappa R_{31}) + J_0(\kappa R_{34})\} \cdot \left[\frac{\cosh(\kappa\tau)}{\sinh(\kappa\tau)} - 1\right] \mathrm{d}\kappa \\ f_2(R_{ij}, \tau) &= \int_0^\infty \{J_0(\kappa R_{21}) - J_0(\kappa R_{24}) - J_0(\kappa R_{31}) + J_0(\kappa R_{34})\}\,\mathrm{d}\kappa \end{aligned}$$

To simplify $f_2(R_{ij}, \tau)$ the following identity can be used

$$\int_0^\infty J_0(\kappa r)\,\mathrm{d}\kappa = \frac{1}{r},$$

and therefore

$$f_2(R_{ij}, \tau) = \frac{1}{R_{21}} - \frac{1}{R_{24}} - \frac{1}{R_{31}} + \frac{1}{R_{34}}. \tag{4}$$

To simplify $f_1(R_{ij}, \tau)$ the following two identities can be used

$$\begin{aligned} \left[\frac{\cosh(\kappa\tau)}{\sinh(\kappa\tau)} - 1\right] &= \frac{2}{-1 + e^{2\kappa\tau}} = 2\sum_{h=1}^\infty e^{-2h\tau\kappa} \\ \int_0^\infty J_0(\kappa r)\,e^{-\beta\kappa}\,\mathrm{d}\kappa &= \frac{1}{\sqrt{r^2 + \beta^2}} \end{aligned}$$

resulting in

$$f_1(R_{ij}, \tau) = 2\sum_{h=1}^\infty \left[\frac{1}{\sqrt{R_{21}^2 + (2h\tau)^2}} - \frac{1}{\sqrt{R_{24}^2 + (2h\tau)^2}} - \frac{1}{\sqrt{R_{31}^2 + (2h\tau)^2}} + \frac{1}{\sqrt{R_{34}^2 + (2h\tau)^2}}\right] \tag{5}$$

In order to handle all the boundary conditions, the concept of mirror currents will be used. Before starting the description, a few more abbreviations will be introduced:

$$\Lambda = L/s; \qquad \lambda_\pm = \ell_\pm/s; \qquad \mu_\pm = m_\pm/s,$$

where "+" refers to I_+ and "−" to I_-.

Fig. 2 shows the original four-point arrangement, here as equidistant inline arrangement ($s - s - s$), together we all mirror currents (here up to 2$^{\text{nd}}$ order). In principle one has to take *all* orders of mirror currents, resulting in an infinite filling of the plane. As can be seen, this tiling of mirror currents can be arranged into 4 sub-lattices, depicted with the colors: green, red, blue, and yellow.

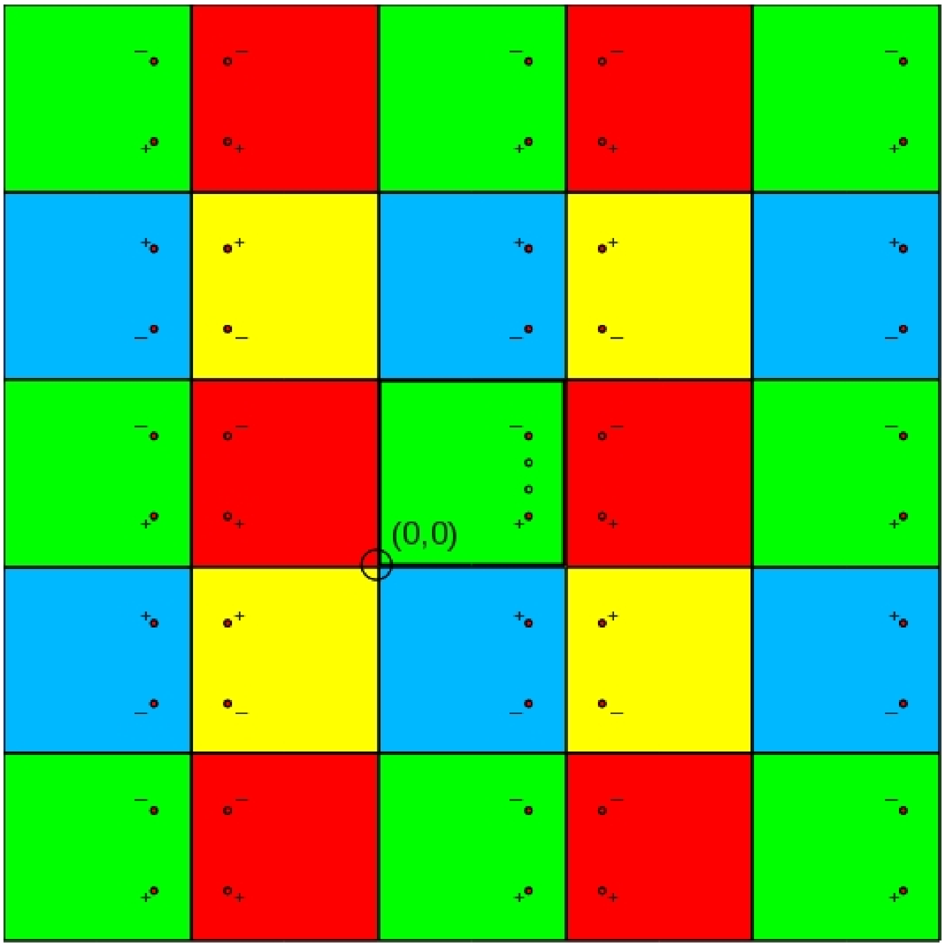

Figure 2: mirror currents

The potential points V_2 and V_3 have the coordinates:

$$\begin{aligned} V_2 &: \vec{r}_2 = (x_2, y_2) \\ V_3 &: \vec{r}_3 = (x_3, y_3) \end{aligned}$$

The 0$^{\text{th}}$ order current points (*i.e.* the real ones), have the coordinates:

$$\begin{aligned} I_+ &: \vec{r}_1 = (x_1, y_1) \\ I_- &: \vec{r}_4 = (x_4, y_4) \end{aligned}$$

The different sub-lattices have therefore the following coordinates:

★ Sub-lattice green:

$$\vec{r}^{\,n,m}_{1,4;g} = \vec{r}_{1,4} + 2n\Lambda\hat{e}_x + 2m\Lambda\hat{e}_y,$$

and therefore

$$\begin{aligned} R^{n,m}_{21;g} &= |\vec{r}_2 - \vec{r}^{\,n,m}_{1;g}| = \sqrt{(x_2 - x_1 - 2n\Lambda)^2 + (y_2 - y_1 - 2m\Lambda)^2} \\ R^{n,m}_{24;g} &= |\vec{r}_2 - \vec{r}^{\,n,m}_{4;g}| = \sqrt{(x_2 - x_4 - 2n\Lambda)^2 + (y_2 - y_4 - 2m\Lambda)^2} \end{aligned}$$

★ Sub-lattice red:

$$\vec{r}^{\,n,m}_{1,4;r} = \vec{r}_{1,4} + 2(\lambda_\pm + n\Lambda)\hat{e}_x + 2m\Lambda\hat{e}_y,$$

and therefore

$$\begin{aligned} R^{n,m}_{21;r} &= |\vec{r}_2 - \vec{r}^{\,n,m}_{1;r}| = \sqrt{(x_2 - x_1 - 2[\lambda_+ + n\Lambda])^2 + (y_2 - y_1 - 2m\Lambda)^2} \\ R^{n,m}_{24;r} &= |\vec{r}_2 - \vec{r}^{\,n,m}_{4;r}| = \sqrt{(x_2 - x_4 - 2[\lambda_- + n\Lambda])^2 + (y_2 - y_4 - 2m\Lambda)^2} \end{aligned}$$

★ Sub-lattice blue:

$$\begin{aligned}\vec{r}_{1;b}^{\,n,m} &= \vec{r}_1 + 2n\Lambda\hat{e}_x + 2[\Lambda - \mu_+ + m\Lambda]\hat{e}_y \\ \vec{r}_{4;b}^{\,n,m} &= \vec{r}_4 + 2n\Lambda\hat{e}_x + 2[\mu_- + m\Lambda]\hat{e}_y\end{aligned}$$

and therefore

$$\begin{aligned}R_{21;b}^{n,m} &= |\vec{r}_2 - \vec{r}_{1;b}^{\,n,m}| = \sqrt{(x_2 - x_1 - 2n\Lambda)^2 + (y_2 - y_1 - 2[(1+m)\Lambda - \mu_+])^2} \\ R_{24;b}^{n,m} &= |\vec{r}_2 - \vec{r}_{4;b}^{\,n,m}| = \sqrt{(x_2 - x_4 - 2n\Lambda)^2 + (y_2 - y_4 - 2[m\Lambda + \mu_-])^2}\end{aligned}$$

★ Sub-lattice yellow:

$$\begin{aligned}\vec{r}_{1;y}^{\,n,m} &= \vec{r}_1 + 2[\lambda_+ + n\Lambda]\hat{e}_x + 2[\Lambda - \mu_+ + m\Lambda]\hat{e}_y \\ \vec{r}_{4;y}^{\,n,m} &= \vec{r}_4 + 2[\lambda_- + n\Lambda]\hat{e}_x + 2[\mu_- + m\Lambda]\hat{e}_y\end{aligned}$$

and therefore

$$\begin{aligned}R_{21;y}^{n,m} &= |\vec{r}_2 - \vec{r}_{1;y}^{\,n,m}| = \sqrt{(x_2 - x_1 - 2[\lambda_+ + n\Lambda])^2 + (y_2 - y_1 - 2[(1+m)\Lambda - \mu_+])^2} \\ R_{24;y}^{n,m} &= |\vec{r}_2 - \vec{r}_{4;y}^{\,n,m}| = \sqrt{(x_2 - x_4 - 2[\lambda_- + n\Lambda])^2 + (y_2 - y_4 - 2[m\Lambda + \mu_-])^2}\end{aligned}$$

To calculate $f(R_{ij}, \tau)$ in all orders, R_{ij} in Eqs.(4)&(5) have to be considered as function of n and m as well. $f(R_{ij}, \tau)$ is hence a sum over all the mirror current tiles

$$f(R_{ij}, \tau) = \sum_{\text{over all tiles}} f(R_{ij}^{n,m}, \tau).$$

There is a small complication here; since in each order one has to sum over a square, the different sub-lattices do *not* run over the exactly same (n, m)-indices range. The summation ranges are listed below.

• green summation range (see also Fig. 2):

$$\begin{aligned}n &= -N \ldots + N \\ m &= -N \ldots + N\end{aligned}$$

• red summation range:

$$\begin{aligned}n &= -N \ldots + (N-1) \\ m &= -N \ldots + N\end{aligned}$$

• blue summation range:

$$\begin{aligned}n &= -N \ldots + N \\ m &= -N \ldots + (N-1)\end{aligned}$$

• yellow summation range:

$$\begin{aligned}n &= -N \ldots + (N-1) \\ m &= -N \ldots + (N-1)\end{aligned}$$

Fig. 3 shows up to which order to summation is needed until the geometry factor $1/f(R_{ij}, \tau)$ converges. Typically $N = 6$ is good enough. The results shown here show the same trend as Fig.6 in B.M. Wojek's memo.

From a fit to the data shown in Fig. 4 the resistance to resistivty conversion factor for a thin homogeneous film on a square substrate ($L \times L = 1 \times 1\,\text{cm}^2$) is obtained.

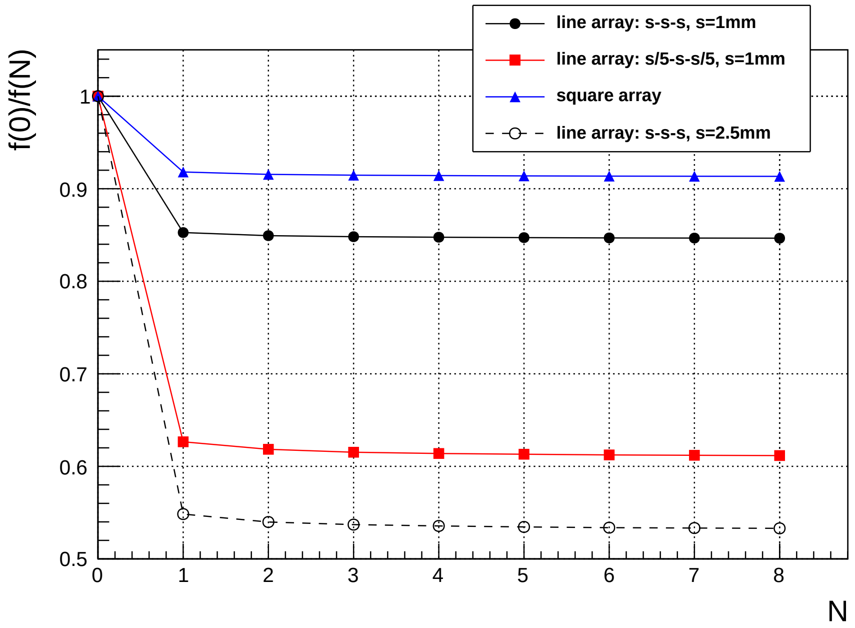

Figure 3: Geometry factor $1/f(R_{ij}, \tau)$ as function of the orders of image currents. The parameters are (thickness always 50 nm, substrate size $L = 10$ mm): (i) line array $s - s - s$, $s = 1$ mm: $\vec{r}_1 = (8\,\text{mm}, 2.5\,\text{mm})$, $\vec{r}_2 = (8\,\text{mm}, 4.5\,\text{mm})$, $\vec{r}_3 = (8\,\text{mm}, 5.5\,\text{mm})$, $\vec{r}_4 = (8\,\text{mm}, 6.5\,\text{mm})$. (ii) line array $s/5 - s - s/5$, $s = 5$ mm: $\vec{r}_1 = (8\,\text{mm}, 1.5\,\text{mm})$, $\vec{r}_2 = (8\,\text{mm}, 2.5\,\text{mm})$, $\vec{r}_3 = (8\,\text{mm}, 7.5\,\text{mm})$, $\vec{r}_4 = (8\,\text{mm}, 8.5\,\text{mm})$. (iii) square array, $s = 1$ mm: $\vec{r}_1 = (7\,\text{mm}, 4.5\,\text{mm})$, $\vec{r}_2 = (8\,\text{mm}, 4.5\,\text{mm})$, $\vec{r}_3 = (8\,\text{mm}, 5.5\,\text{mm})$, $\vec{r}_4 = (7\,\text{mm}, 5.5\,\text{mm})$. (iv) line array $s - s - s$, $s = 2.5$ mm: $\vec{r}_1 = (8\,\text{mm}, 1.25\,\text{mm})$, $\vec{r}_2 = (8\,\text{mm}, 3.75\,\text{mm})$, $\vec{r}_3 = (8\,\text{mm}, 6.25\,\text{mm})$, $\vec{r}_4 = (8\,\text{mm}, 8.75\,\text{mm})$.

$$
\begin{aligned}
\rho\,[\text{m}\Omega\text{cm}] &= 3.85 \times 10^{-4} \cdot t\,[\text{nm}] \cdot R[\Omega], && \text{for the situation (1)} \\
&= 8.30 \times 10^{-4} \cdot t\,[\text{nm}] \cdot R[\Omega], && \text{for the situation (2)} \\
&= 1.08 \times 10^{-4} \cdot t\,[\text{nm}] \cdot R[\Omega], && \text{for the situation (3)} \\
&= 2.45 \times 10^{-4} \cdot t\,[\text{nm}] \cdot R[\Omega], && \text{for the situation (4)} \\
&= 2.01 \times 10^{-4} \cdot t\,[\text{nm}] \cdot R[\Omega], && \text{for the situation (5)} \\
&= 1.74 \times 10^{-4} \cdot t\,[\text{nm}] \cdot R[\Omega], && \text{for the situation (6)}
\end{aligned}
$$

(1) see Fig. 5 (a), $s = r_{23} = 1\,\text{mm}$, $\ell = 2\,\text{mm}$, $m = 3.5\,\text{mm}$

(2) see Fig. 5 (b), $s = r_{23} = 1\,\text{mm}$, $\ell = 2\,\text{mm}$, $m = 4.5\,\text{mm}$

(3) see Fig. 5 (c), $s = r_{23} = 5\,\text{mm}$, $\ell = 2\,\text{mm}$, $m = 1.5\,\text{mm}$, $\alpha = 5$

(4) see Fig. 5 (a), $s = r_{23} = 2.5\,\text{mm}$, $\ell = 2\,\text{mm}$, $m = 1.25\,\text{mm}$

(5) see Fig. 5 (a), $s = r_{23} = 2.5\,\text{mm}$, $\ell = 1\,\text{mm}$, $m = 1.25\,\text{mm}$

(6) see Fig. 5 (c), $s = r_{23} = 2.5\,\text{mm}$, $\ell = 2\,\text{mm}$, $m = 2.25\,\text{mm}$, $\alpha = 5/3$

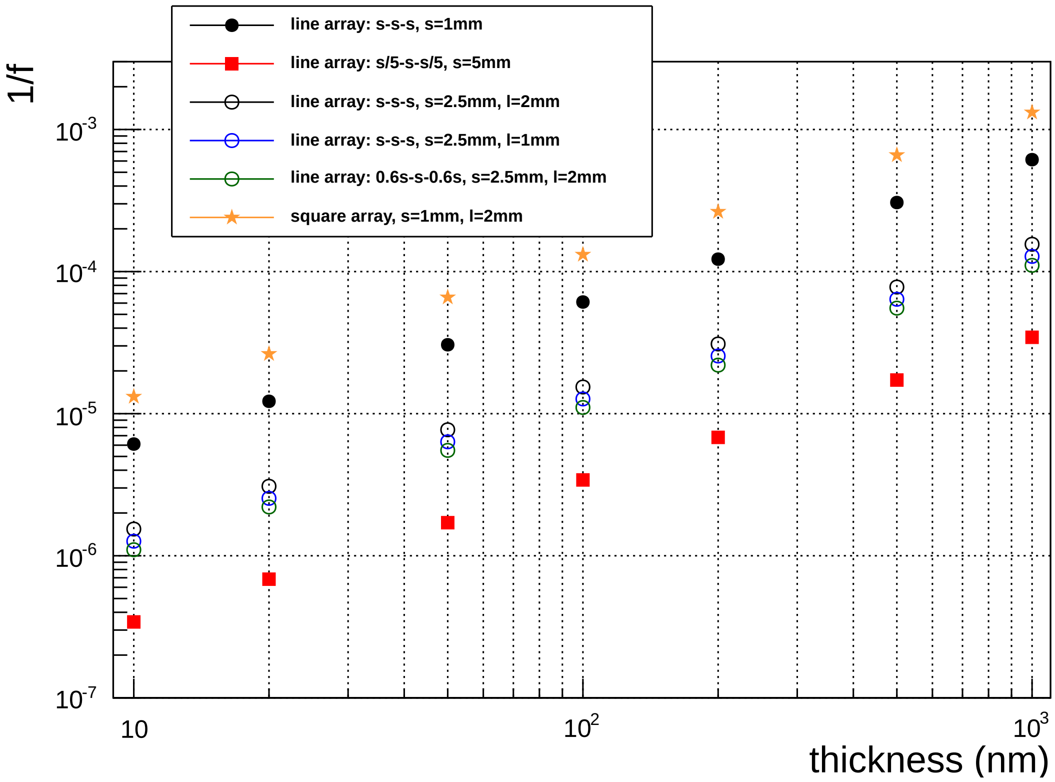

Figure 4: The geometry factor $1/f$ for the arrangements given above. Calculated in order $N = 6$.

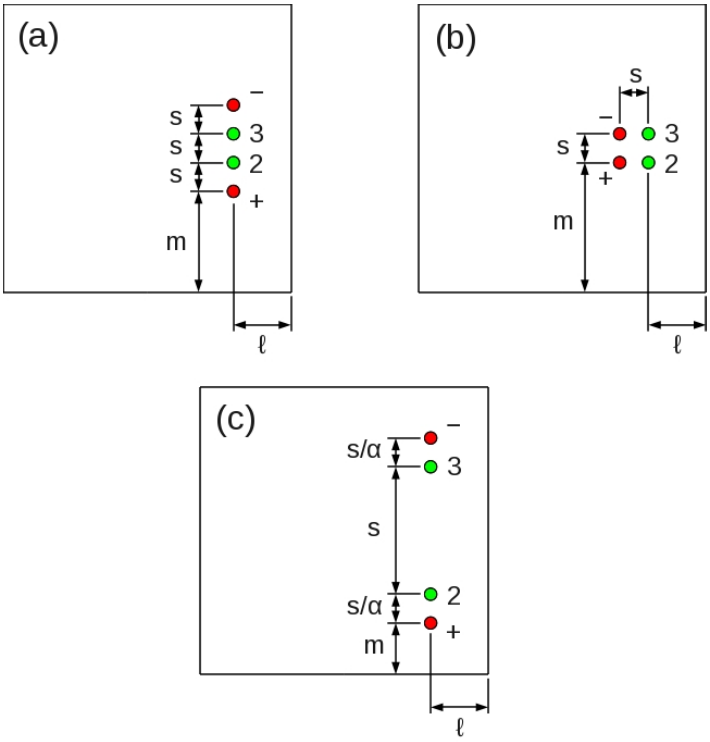

Figure 5: Pin arrangements as used for the above calculations.

Comments to B.M. Wojek's Memo: "Four-Point Resistivity Correction Factors for Thin Films"

1. All the graphs shown in B.M. Wojek's memo seem to be OK.
2. The results obtained by Eqs. (27) to (30) are consistent with those presented in this memorandum. However, the use of "vertical parameters" to represent "horizontal distances" is confusing.
3. The variable s is defined "locally" for each geometry but it is not used coherently throughout the memorandum in a "global" sense. The most coherent definition in terms of a generalization is given by $s = |\vec{r}_2 - \vec{r}_3|$. Therefore, the so-called $s - 5s - s$ geometry would better be treated as $s/5 - s - s/5$. The way B.M. Wojek's memo is written this is, unfortunately not possible.

Program which can be used to Calculate the Thin Film Correction Terms for Arbitrary Pin Arrangement

Under `<what_ever>/analysis/root/macros` you will find a file `resistivity.C` which consists out of two parts: (i) a class called `PResistivity` which does all the calculations needed, and a simple function `resistivity` which can be used to feed the parameters and start the calculation. In order to do so, start `ROOT`, then follow the Instructions below.

```
[nemu@pcXXXX macros]$ root -l
root [0] .L resistivity.C++
Info in <TUnixSystem::ACLiC>: creating shared library /home/nemu/analysis/root/macros/./resistivity_C.so
root [1] resistivity(6, 1.0, 50e-9, 10e-3, 8e-3, 1.25e-3, 8e-3, 8.75e-3, 8e-3, 3.75e-3, 8e-3, 6.25e-3)

s=0.0025

normalized size =4
fLambdaP =0.8
fLambdaM =0.8
fMuP     =0.5
fMuM     =0.5
---
R21g(0,0)=1
R24g(0,0)=2
R31g(0,0)=2
R34g(0,0)=1

N=0: 1/F1=1.44287e-05

N=1: 67692.8, 28296.6, 19510.3, 10871.4,
N=1: 1/F1=7.91321e-06, result0/result=0.548437

N=2: 67139.4, 28302.3, 21368.9, 11567.7,
N=2: 1/F1=7.78948e-06, result0/result=0.539861
```

The function `resistivity` has the following arguments:

```
Double_t resistivity(UInt_t order,        // up to which order the correction shall be calculated
                     Double_t resistance, // resistance given in (Ohm)
                     Double_t thickness,  // film thickness given in (m)
                     Double_t L,          // size of the square substrate in (m)
                     Double_t xIp,        // I_+ x-coordinate
                     Double_t yIp,        // I_+ y-coordinate
                     Double_t xIm,        // I_- x-coordinate
                     Double_t yIm,        // I_- y-coordinate
                     Double_t xV2,        // V_2 x-coordinate
                     Double_t yV2,        // V_2 y-coordinate
                     Double_t xV3,        // V_3 x-coordinate
                     Double_t yV3)        // V_3 y-coordinate
```